CS Basics
1) Bases 2, 8, 16, etc.

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E.Benoist, C.Fuhrer, Ch. Grothoff, L. Ith, P.Mainini | BFH-TI

Bases

- **Bases**
- **Hexadecimal**
  - Conversions
  - Arithmetic in Hexadecimal
- **Binary**
  - Hex as shorthand for binary
- **Octal**
  - Octal as shorthand for binary
- **Conclusion**

Bases

We count in Base 10

Signs used for counting

- $1, 2, 3, 4, 5, 6, 7, 8, 9$, and 0
- A number is a list of signs: 123 means $1 \times 100 + 2 \times 10 + 3$

Other bases were used over the time

- Base 12 (e.g. for hours)
- Base 60 (e.g. for minutes):
  50 minutes and 33 seconds is a time counted in base 60;
  it is $50 \times 60 + 33 = 3033$ seconds
  BTW: $60 = 5 \times 12$

In the 70's other bases were used to teach counting

- "Modern mathematics"

Essence of a number base

We use *columnar systems*: the position of a number represents its value

- in all bases 10 represents the base
- Number in column 0 is multiplied by $base^0 = 1$
- Number in column 1 is multiplied by $base^1 = 10_{base}$
- Number in column 2 is multiplied by $base^2 = 100_{base}$
- Number in column 3 is multiplied by $base^3 = 1000_{base}$

Hexadecimal

Hexadecimal

Hexadecimal = Base 16

- Is the important base for programmers

Digits

- 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, *A (10)*, *B (11)*, *C (12)*, *D (13)*, *E (14)*, *F (15)*
- 16 is the base and therefore written 10 in hexadecimal

Counting

- 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 1A, 1B, 1C, 1D, 1E, 1F, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 2A, 2B, 2C, 2D, 2E, 2F, 30, ...

Notation

- Hexadecimal numbers are denoted by preceding them with “*0x...*”.
- So 1 becomes 0x1, 2F becomes 0x2F
- Upper and lower case (0x2f) are equally valid

Hexadecimal table I

$0x0$	zero	0
$0x1$	one	1
$0x2$	two	2
$0x3$	three	3
$0x4$	four	4
$0x5$	five	5
$0x6$	six	6
$0x7$	seven	7
$0x8$	eight	8
$0x9$	nine	9
$0xA$	A	10
$0xB$	B	11
$0xC$	C	12
$0xD$	D	13
$0xE$	E	14
$0xF$	F	15
$0x10$	One -oh hexadecimal	16

$0x11$	One -one hexadecimal	17
$0x12$	One -two hex.	18
$0x13$	One -three hex.	19
$0x14$	One -four hex.	20
$0x15$	One -five hex.	21
$0x16$	One -six hex.	22
$0x17$	One -seven hex.	23
$0x18$	One -eight hex.	24
$0x19$	One -nine hex.	25
$0x1A$	One -A hex.	26
$0x1B$	One -B hex.	27
$0x1C$	One -C hex.	28
$0x1D$	One -D hex.	29
$0x1E$	One -E hex.	30
$0x1F$	One -F hex.	31
$0x20$	Two -oh hex.	32

Table of powers of 16

Hexadecimal uses powers of 16

$0x1$	16^0	1
$0x10$	16^1	16
$0x100$	16^2	256
$0x1\ 000$	16^3	4 096
$0x10\ 000$	16^4	65 536
$0x100\ 000$	16^5	1 048 576
$0x1\ 000\ 000$	16^6	16 777 216

Anatomy of a number

Let us evaluate the number $0x3C0A9$

$$\begin{array}{rrrrrr} & & & & & 9 \\ & & & & A & 0 \\ & & & 0 & 0 & 0 \\ & & C & 0 & 0 & 0 \\ + & 3 & 0 & 0 & 0 & 0 \\ \hline & 3 & C & 0 & A & 9 \end{array}$$

$$3 \times 65\,536 + 12 \times 4096 + 0 \times 256 + 10 \times 16 + 9 \times 1$$

$$196\,608 + 49152 + 0 + 160 + 9 = 245\,929$$

From hexadecimal to decimal

Method

- Compute the value of each column and add the results

Decimal value of $0x7A2$

- add 2 (for the 2 in column 0)
- add 10×16 (for the A in column 1)
- add 7×256 (for the 7 in column 2)

Other example

Value of $0xC6F0DB$

- $B \times 1 = 11$
- $D \times 16 = 13 \times 16 = 208$
- $0 \times 256 = 0$
- $F \times 4096 = 15 \times 4096 = 61\,440$
- $6 \times 65\,536 = 393\,216$
- $C \times 1\,048\,576 = 12 \times 1\,048\,576 = 12\,582\,912$

Total = 13 037 787

From decimal to hexadecimal

We want to write 449 in hex.

- Find the largest hex column value that is contained at least once in 449: 4096 is to large, 256 fits well
- Find how many times 256 fits into 449
  - $449/256 = 1.7539$ so **1** is the leftmost hex digit
  - Subtract 1×256 from 449, we obtain 193
- The next power of 16 is 16 itself; how many times does 16 fit into 193?
  - $193/16 = 12.0625$, so **C** is the next hex digit
  - $193 - 12 * 16 = 1$; the remainder is 1 and the next value of 16 (i.e. 16^0) is **1**, so the last hex digit is **1**

The hex value of 449 is $0x1C1$

Arithmetic in hexadecimal

You need to do arithmetic directly in Hex

- Conversion to decimal and back takes too much time
- Example: to add $0xC$ and $0xF$
  - $0xC$ is 12, $0xF$ is 15, $0xC + 0xF$ is 27
  - then we convert 27 back into hex: $0x1B$

Need to learn additions by heart

- Use flash cards for instance

Columns and carries

Method for adding hex numbers

Add each column starting from the right and carry into the next column anytime the result exceeds $0xF$

$$
\begin{array}{rcccccc}
 & \color{blue}{1} & & & & \color{blue}{1} & \\
 & 2 & F & 3 & 1 & A & D \\
+ & 9 & 6 & B & A & 0 & 7 \\
\hline
 & C & 5 & E & B & B & 4
\end{array}
$$

Note: The carry never exceeds 1

Subtraction and borrows

We have to mentally reverse

- if $E + 6 = 0x14$ then $0x14 - 6 = E$

We have to subtract column by column

- Start from right

$$\begin{array}{ccccc} & F & 7 & 6 & C \\ - & A & 0 & 5 & B \\ \hline & 5 & 7 & 1 & 1 \end{array}$$

Borrows

Need for borrows if the value to subtract is larger than the one we subtract from

- $9 - A = ???$

$$\begin{array}{cccc} & 9 & 2 \\ - & 4 & F \\ \hline & ? & ? \end{array}$$

- We need to add `0x10` (i.e. 16_{10}) to the number for the subtraction to be possible

$$\begin{array}{cccc} & 9 & 2 \\ - & 4_1 & F \\ \hline & 4 & 3 \end{array}$$

Borrows across multiple columns

We may have to transfer the borrow across more than one column

- $$\begin{array}{rcccc} & F & 0 & 0 & 0 \\ - & 3 & B & 6 & C \\ \hline & ? & ? & ? & ? \end{array}$$

- $$\begin{array}{rcccc} & F & 0 & 0 & 0 \\ - & 3 & B & 6_1 & C \\ \hline & ? & ? & ? & 4 \end{array}$$

- $$\begin{array}{rcccc} & F & 0 & 0 & 0 \\ - & 3 & B_1 & 6_1 & C \\ \hline & ? & ? & 9 & 4 \end{array}$$

Borrows across multiple columns

- $$\begin{array}{ccccc} & F & 0 & 0 & 0 \\ - & 3_1 & B_1 & 6_1 & C \\ \hline & ? & 4 & 9 & 4 \end{array}$$

- $$\begin{array}{ccccc} & F & 0 & 0 & 0 \\ - & 3_1 & B_1 & 6_1 & C \\ \hline & B & 4 & 9 & 4 \end{array}$$

Binary

Binary

The base consists only of the two digits 0 and 1 (Base 2)

Each column has a value two times the column to its right

Counting

- 0, 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, ...

Powers of 2

Binary	Power of 2	Decimal
0b1	2^0	1
0b10	2^1	2
0b100	2^2	4
0b1000	2^3	8
0b10000	2^4	16
0b100000	2^5	32
0b1000000	2^6	64
0b10000000	2^7	128
0b100000000	2^8	256
0b1000000000	2^9	512
0b10000000000	2^{10}	1 024
0b100000000000	2^{11}	2 048
0b1000000000000	2^{12}	4 096
0b10000000000000	2^{13}	8 192
0b100000000000000	2^{14}	16 384
0b1000000000000000	2^{15}	32 768
0b10000000000000000	2^{16}	65 536

Notation

Values in binary are denoted with leading "*ob...*" or sometimes with B at the end

$0b110$ means 6

whereas

$0x110$ means 272

and 110 means 110

Notations in scientific books uses subscript

110_2 means 6_{10}

whereas

110_{16} means 272_{10}

and 110_{10} means 110_{10}

However subscript is not generally usable within source code and other files

Why is binary important for computers?

Because “lights are either on or off”

- In an electrical device : voltage is present or not
- Naturally represents 1 or 0

Other machines have been tested with base 3

- 1840 Thomas Fowler built a ternary calculating machine from wood
- 1958 Nikolay Brusentsov (USSR) built the *Setun* computer
- In 1973 he built an enhanced version called *Setun-70*
- In the USA, the *Ternac* was built in 1973

Hex as shorthand for binary

Example: 218

- **in binary:** $0b11011010$
- **in hexadecimal:** $0xDA$

We split: $0b1101\ 0b1010$

- This is 13 and 10 or $0xD$ and $0xA$ respectively
- 218 in hex is $0xDA$

Eight binary digits may be conveniently written as 2 hexadecimal digits

218		decimal
1101	1010	binary
D	A	hex

Hex as shorthand for binary

If we have a 32 digit binary number

$0b11110000000000001111101001101110$

We can split it into groups of 4

$0b1111\ 0b0000\ 0b0000\ 0b0000\ 0b1111\ 0b1010\ 0b0110\ 0b1110$

Each group is represented by one Hex value

$$\begin{array}{cccccccc} 1111 & 0000 & 0000 & 0000 & 1111 & 1010 & 0110 & 1110 \\ F & 0 & 0 & 0 & F & A & 6 & E \end{array}$$

The hex equivalent is $0xF000FA6E$

Octal

★ Octal

Counting in octal

- 0, 1, 2, 3, 4, 5, 6, 7, 10
- We do not use 8 and 9 anymore
- 10 means 8
- 11 means 9
- 12 means 10

Octal uses base 8

- So the 8 does not exist!
- 27 octal means $2 \times 8 + 7$ (=23 in decimal)

Octal table

0	zero	0
1	one	1
2	two	2
3	three	3
4	four	4
5	five	5
6	six	6
7	seven	7

10	ten octal	8
11	eleven octal	9
12	twelve octal	10
13	thirteen octal	11
14	fourteen oct.	12
15	fifteen oct.	13
16	sixteen oct.	14
17	seventeen oct.	15
20	twenty oct.	16

The octal numbers

Value of a number depends on its column

- A number in the unit column (column number 0) is just its value
- 7 octal means 7

Column number one is multiplied by 8

- 10 octal means 8
- 20 octal means 16
- 70 octal means $7 \times 8 = 56$

Column number two is multiplied by 64

- 100 octal means 64

★ The powers of 8

$1\text{ octal} = 8^0 = 1$

$10_8 = 8^1 = 8$

$100_8 = 8^2 = 64$

$1\ 000_8 = 8^3 = 512$

$10\ 000_8 = 8^4 = 4\ 096$

$100\ 000_8 = 8^5 = 32\ 768$

★ Converting from octal into decimal

Suppose we have the number 76225_8

- $76225_8 = 70000_8 + 6000_8 + 200_8 + 20_8 + 5_8$
- $5_8 = 5 \times 1 = 5$
- $20_8 = 2 \times 10_8 = 2 \times 8^1 = 16$
- $200_8 = 2 \times 100_8 = 2 \times 8^2 = 128$
- $6000_8 = 6 \times 1000_8 = 6 \times 8^3 = 3072$
- $70000_8 = 7 \times 10000_8 = 7 \times 8^4 = 28672$

- $76225_8 = 28672 + 3072 + 128 + 16 + 5 = 31893$

★ Octal as shorthand for binary

Example: 218

- **in binary:** $0b11011010$
- **in octal:** 0332

We split: $0b11\ 0b011\ 0b010$

Three binary digits are represented with one octal digit

	218		decimal
11	011	010	binary
3	3	2	octal

Conclusion

Conclusion

Computers work with binary numbers only

- Binary notation is too cumbersome.
- In general, we use hexadecimal to represent binary values.
- Octal is rarely used today.

You should be familiar with hex notation

- Used a lot in upcoming assembler programming, C programming. and also in following courses.
- Only one solution: do the exercises!