CS Basics 2)
Encoding Numbers

Fall Term 2023-24

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Contents

Binary vs. Text Files

Binary vs. Text Files

Different file formats (examples)

- Images: `jpeg`, `gif`, `png`, `svg`, …
- Documents: `pdf`, `doc`, `ppt`, `xls`, …
- Executables (Programs): `exe`, `dll`, `so`, `class`, …
- Plain text: `txt`, `html`, `tex`, …
- Source code: `asm`, `c`, `cpp`, `java`, …

Two primary types

- *Binary* files (images, documents, executables, …)
- *Text files* (plain text, source code, …)

Distinction not always strict

- `svg` is a format for vector graphics based on XML and can thus be considered a text file
- Office documents are often also a form of compressed XML
- …

Text Files

Different character encodings

- 7 bit / US-ASCII
- ISO-8859-*
- UTF-8
- …

How to work with text files?

- Using any text editor or IDE
- vim, emacs, Notepad (windows), gEdit, kate, sublime, atom, …
- Eclipse, NetBeans, IntelliJ, kDevelop, …

Binary Files I

A lot of different file formats

- Open and proprietary
- Encoding and structure depend on type!

Executable files (Programs)

- Windows: `exe`
- GNU/Linux: `elf` (32 bits), `elf64` (64-bit)
- Contain machine instructions and data
- Can be analyzed with a *decompiler* or *disassembler*

Images

- `jpeg`: format family, lossy compression
- `gif`: 8-bit color, animations possible
- `png`: 32-bit color, no animations, modern lossless compression (good for screenshots)
- `svg`: vector graphics, not a binary file type (see Slide 4)
- Can be manipulated with libraries and image editors

Binary Files II

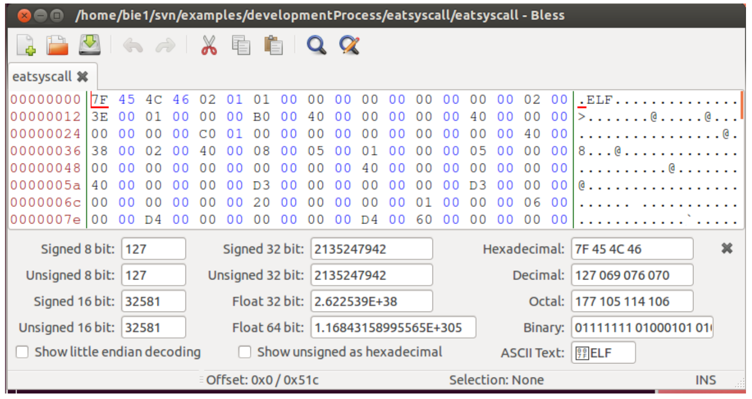
How to analyze any type of binary data?

- Using a *hex editor*
- Editors: Bless, xxd, …

Bless functionality

- Display binary and textual representation at the same time
- Various conversions to different representations
- Can read and edit any file – also executable files

Binary Files III


/home/bie1/svn/examples/developmentProcess/eatsyscall/eatsyscall - Bless
eatsyscall

```
00000000 7F 45 4C 46 02 01 01 00 00 00 00 00 00 00 00 00 02 00 .ELF..............
00000012 3E 00 01 00 00 00 B0 00 40 00 00 00 00 00 40 00 00 00 >.......@.....@...
00000024 00 00 00 00 C0 01 00 00 00 00 00 00 00 00 00 00 40 00 ................@.
00000036 38 00 02 00 40 00 08 00 05 00 01 00 00 00 05 00 00 00 8...@.............
00000048 00 00 00 00 00 00 00 00 00 00 40 00 00 00 00 00 00 00 ..........@.......
0000005a 40 00 00 00 00 00 D3 00 00 00 00 00 00 00 D3 00 00 00 @.................
0000006c 00 00 00 00 00 00 20 00 00 00 00 00 01 00 00 00 06 00 ......  ..........
0000007e 00 00 D4 00 00 00 00 00 00 00 D4 00 60 00 00 00 00 00 ..............`...
```

Signed 8 bit: 127 | Signed 32 bit: 2135247942 | Hexadecimal: 7F 45 4C 46
Unsigned 8 bit: 127 | Unsigned 32 bit: 2135247942 | Decimal: 127 069 076 070
Signed 16 bit: 32581 | Float 32 bit: 2.622539E+38 | Octal: 177 105 114 106
Unsigned 16 bit: 32581 | Float 64 bit: 1.16843158995565E+305 | Binary: 01111111 01000101 01
☐ Show little endian decoding | ☐ Show unsigned as hexadecimal | ASCII Text: ELF

Offset: 0x0 / 0x51c Selection: None INS

Binary Files IV

Data is ultimately always encoded in binary

Example

- Capital letter “`S`” is encoded with `0x53` (ASCII/UTF-8)
- For the computer, this is a set of 8 bits: `01010011`
- Can also be part of a different number
  - `0x53` may be interpreted as decimal 83
  - `0x53_61` may be interpreted as the decimal 21’345
  - `0x53_61_6D_0A` may be interpreted as the decimal 1’398’893’834
  - `0x53_61_6D_0A_77_61_73_0A` may be interpreted as the floating point number $4.54365038640977.10^{93}$
- But: This pattern can also mean anything else
  - Can be part of an instruction
  - Can be any data (object, address, ...)
  - ...

Endianness

Endianness

Having multiple bytes of data...

- Example: `0x53 0x61 0x6D 0x0A 0x77 0x61 0x73 0x0A`
- Can be read/interpreted left-to-right (like European languages)

```
0x53 0x61 0x6D 0x0A 0x77 0x61 0x73 0x0A
0    1    2    3    4    5    6    7
```

- ...but can also be read/interpreted right-to-left (like in Arabic)

```
0x0A 0x73 0x61 0x77 0x0A 0x6D 0x61 0x53
7    6    5    4    3    2    1    0
```

- The same data — but different meaning!

Which number does the above data represent?

- 0x53616D0A7761730A ?
- 0x0A7361770A6D6153 ?

Big Endian vs. Little Endian

Big Endian

- Data is stored with the *most significant byte (MSB)* first
- Example: Number 0x20A1 is stored as follows

```
0x20 0xA1
0    1
```

Little Endian

- Data is stored with the *least significant byte (LSB)* first
- Example: Number 0x20A1 is stored as follows

```
0xA1 0x20
0    1
```

Endianness is dependent of CPU architecture

- *Intel x86 uses little endian*
- Other architectures may use big endian
- Some CPUs even support both

Big Endian vs. Little Endian

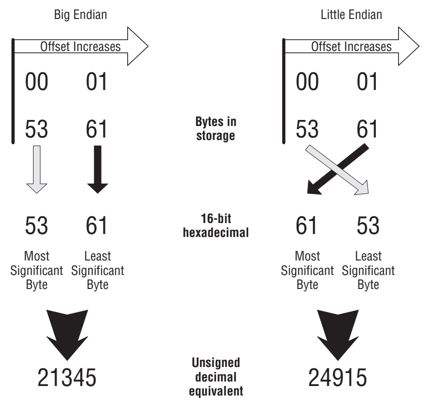


Figure 5-5: Big endian vs. little endian for a 16-bit value

Negative Numbers

How to Represent Negative Numbers?

So far, we have seen positive numbers

- Example: 15
- In memory: 0x0F (8 bit) or 0x0000000F (32 bit)

What about negative numbers?

- Example: -15
- In memory: ???

Idea: Sign Bit

What if we use one bit for the sign?

- Taking the first bit as sign bit...
- 15 is still 0x000F (16 bit)
- -15 would be 0x800F

Problems

- Two representations of 0 exist: 0x0000 and 0x8000
- Standard (bit-level) addition does not work:

  15 + (-15) = 0x000F + 0x800F = 0x801E = -30

One's Complement

One's Complement

- The negative number is obtained by inverting all the bits
- 15 = 0x000F
- 15 = 0b0000 0000 0000 1111
- -15 is represented by 0b1111 1111 1111 0000
- -15 = 0xFFF0

Problems

- We *still* have two representations of 0: 0x0000 and 0xFFFF
- Addition *still* does not work:

  15 = 0x000F and -5 = 0xFFFA. The sum is 0x10009 = 9, if we ignore the overflow

Two's Complement

Negative numbers must respect arithmetic

- $15 + (-5) = 10$
- Idea: $-X$ is represented by $2^n - X$, with n being the *bit-width*
- $X + (-X) = X + (2^n - X) = 0$ *with overflow*

Example 1

- Representation of $-\mathbf{1}$ in 16 bits
- $2^{16} - \mathbf{1} = 2^{15} + 2^{14} + 2^{13} + 2^{12} + 2^{11} + \cdots + 2^1 + 2^0$
- -1 is written 0xFFFF
- $1 + (-1) = 0x10000$ (overflow!)

Example 2

- Representation of $-\mathbf{20}$ in 16 bits
- $2^{16} - \mathbf{20} = 65516 = \ldots$
- …hard to compute

Two's Complement

Method for easy computation of two's complement

- Take the binary representation of X
- Invert all the bits of X
- Add 1

Example 2 again

- Want: -20
- 20 = 0b0000 0000 0001 0100
- …inverting all the bits
- 0b1111 1111 1110 1011
- …adding 1
- 0b1111 1111 1110 1100 = 0xFFEC

Arithmetic works!

- $15 + (-5) = 10$
- 0b0000 1111 + 0b1111 1011 = 0b1 0000 1010
- Ignoring overflow: 0b0000 1010 = 10

Introduction to Floating Point

Floating Point Arithmetic

In the early days, CPUs where only able to handle integer arithmetic.

- However, for scientific and also everyday purposes, *real numbers* ($\mathbb{R}$) are required.
- The data types for working with real numbers have special properties that every programmer must know.

Representation of real numbers and *floating point arithmetic* is standardized in the *IEEE 754 standard*, which is supported by most programming languages and CPUs nowadays.

Fixed Point Numbers

How to represent the decimal number 543.125 in *fixed point* arithmetic? Recall: Every number in base b can be written as

$$x = \sum_{i=-\infty}^{\infty} d_i b^i$$

where d_i are the respective digits of the number and b is the base.

This way, the number 543.125 may be written as the sum:

$$543.125 = 5 \cdot 10^2 + 4 \cdot 10^1 + 3 \cdot 10^0 + 1 \cdot 10^{-1} + 2 \cdot 10^{-2} + 5 \cdot 10^{-3}$$

Binary Fixed Point Numbers

As the number 543.125 exists *independently* of any base, it must be possible to obtain a binary representation. We proceed as follows:

1. First, the integral part of the number is converted to binary

   $543 = 0\mathrm{b}10\,0001\,1111$
2. Then, the fractional part of the number is converted as well

   $0.125 = 0\mathrm{b}0.001$

The whole, fixed point binary number is then

$543.125 = 0\mathrm{b}10\,0001\,1111.001$

Converting to Fixed Point Binary I

First, we convert the *integer* part as usual, using division by 2 with remainder:

Example: 543

```
543  |  1
271  |  1
135  |  1
67   |  1
33   |  1
16   |  0
8    |  0
4    |  0
2    |  0
1    |  1
```

$543 = 0\mathrm{b}10\,0001\,1111$

Converting to Fixed Point Binary II

To convert the *fractional* (or *decimal*) part, we do the opposite and multiply by two until we have no remainder:

Example: 0.125

```
0.125 | 0.
0.25  | 0
0.5   | 0
1     | 1
```

$0.125 = 0\text{b}0.001$

Some Mathematical Considerations

A rational number ($\mathbb{Q}$) is either finite or infinite periodic.

- The property of “rationality” is independent of the base used to represent a number.

A rational number may have a finite representation in one base and a periodic infinite representation in another base.

- Example: the number $\frac{1}{3}$ has an infinite periodic represention in base 10…
- …but is written 0.1_3 in base 3!

Irrational numbers ($\mathbb{R} \setminus \mathbb{Q}$) are infinite and non-periodic.

- They do not have a finite representation *in any* base.

Infinite Periodic Number Example

0.1 is finite in decimal — but not in binary:

```
0.1 | 0.
0.2 | 0
0.4 | 0
0.8 | 0
1.6 | 1 -> 0.6
1.2 | 1 -> 0.2
0.4 | 0
0.8 | 0
1.6 | 1 -> 0.6
1.2 | 1 -> 0.2
...
```

$0.1 = 0\text{b}0.000110011\overline{0011}$

Scientific Notation

- The main drawback of fixed point numbers is that their size depends on their magnitude and their precision.
- In engineering, most of the time, the precision can be reduced as the magnitude increases. This is the concept of *significant digits*.
- On pocket calculators, this is usually known as “scientific notation”. Examples:
  - $1.344 \cdot 10^4$ ($= 13\,440$)
  - $2.342 \cdot 10^{-5}$ ($= 0.000\,023\,42$)
  - $4.430 \cdot 10^0$ ($= 4.430$)
- All theses numbers have 4 significant digits (called the *mantissa*). The *exponent* (10^x) “scales” the numbers.
- With 4 significant digits, the mantissa is always between 1.000 and 9.999.

The IEEE 754 Standard

The IEEE 754 standard used for binary representation of floating point numbers is based on the same idea:

- The mantissa is a number ≥ 1.0 and < 2.0.
- The exponent is now an exponent of 2. It may take positive or negative values.
- Maximum exponent and *precision* (number of significant digits in the mantissa) depends on the encoding format, see Slide 36.
- The standard also introduces special values (negative infinity, positive infinity, not a number, zero).
- It also contains rules for rounding.

IEEE 754, An Example

Example: Encoding the decimal number 0.5 $(= 1.0 \cdot 2^{-1})$ in IEEE 754:[1]

$0.5 =$ `0 0111 1110 0000 0000 0000 0000 0000 000`

- The first bit is the sign bit. Its value is $\mathbf{0}$ for positive numbers and 1 for negative numbers.
- The exponent, in this case $\mathbf{2^{-1}}$, is encoded with a so-called bias.
- The mantissa is $\mathbf{1}$ (not encoded due to the hidden bit).

[1]The number is encoded in single precision format (e.g. `float` in C or Java).

IEEE 754, The Exponent

- The exponent can be positive or negative.
- It is encoded as a signed integer with bias, i.e. without using two's complement!
- The bias is added to the original exponent and corresponds to half the magnitude of the exponent *minus one*.
  For example, if the exponent is 8 bits, its magnitude is $2^8 = 256$ and the bias is $2^{(8-1)} - 1 = 127$.
- Two exponents, 00_{16} and FF_{16}, are reserved to increase precision for very small numbers and for representing special values.

IEEE 754, The Mantissa

- Contains a value in the interval $1 \leq$ mantissa < 2.
- As the mantissa is always ≥ 1, the bit for the 0th power (i.e. 2^0) may be ommitted, increasing precision. This is called the hidden bit. Due to this, the mantissa is 0 in the given example.
- Exception: For denormalized numbers (exponent 0), the mantissa m is $0 < \texttt{mantissa} < 1$! This is used for very small numbers (below 2^{-126}).
- The bits of the mantissa represent the sum of negative powers of 2.

IEEE 754, Values

The following table lists possible values for IEEE 754 encoded numbers:

Exponent	Mantissa	Value	Description
00_{16}	= 0	0	Zero
00_{16}	> 0	$\pm 0.m \cdot 2^{(1-b)}$	Denormalized
01_{16} to FE_{16}	any	$\pm 1.m \cdot 2^{(e-b)}$	Normalized
FF_{16}	= 0	$\pm\infty$	$\pm$ Infinity
FF_{16}	> 0	NaN	Not a Number (NaN)

Remarks:

- e is the exponent, b the bias value.
- There are two possible representations for the value 0 (+0 and -0).
- Exponents $\mathbf{00_{16}}$ and $\mathbf{FF_{16}}$ represent special values!
- All other exponents (01_{16} to FE_{16}) are *normalized* values.

Comments on Special Values

- The value zero needs a special representation as it is not possible to represent it using the standard encoding (see example above).
- Infinity represents numbers whose magnitude cannot be encoded. However, arithmetical operations where one operand is infinity are well defined by IEEE 754 (see next slide).
- Operations where one operand has the value NaN cause an error.
- Warning: Handling of denormalized values is hardware and implementation dependent and may lead to performance issues. This must be considered when using them.

IEEE 754
Operations with Special Values

Operation	Result
$x/\pm\infty$	0
$\pm\infty \cdot \pm\infty$	$\pm\infty$
$\pm$ non zero / 0	$\pm\infty$
$\pm 0/\pm 0$	NaN
$\infty + \infty$	∞
$\infty - \infty$	NaN
$\pm\infty/\pm\infty$	NaN
$\pm\infty \cdot 0$	NaN

IEEE 754 Formats

IEEE 754 (2019) defines three *basic formats* for binary floating point numbers:[2] *Single precision*, *double precision* and *quadruple precision*. They differ only by the number of bits required to store them:

	Sign	Exponent	Mantissa	Bias
Single precision (32 bit)	1	8	23	127
Double precision (64 bit)	1	11	52	1023
Quadruple precision (128 bit)	1	15	112	16383

Note: Due to the hidden bit, precision equals to mantissa $+1$.

[2]Formats with different bit widths, as well as decimal formats (using radix 10) are also specified as non-basic formats and are not treated here.

IEEE 754 Rounding

IEEE 754 has five different rounding modes. The first two round to nearest, the others are *directed roundings*.

- **Round to nearest, ties to even**
  Round to nearest encodable value. If exactly midway, round to the nearest value where the least significant bit of the mantissa is 0.
- **Round to nearest, ties away from zero**
  Round to nearest also, but round up (positive numbers) or down (negative numbers)
- **Round toward 0**
  Rounds to 0 (*truncation*)
- **Round toward** $+\infty$
  Always round *up*.
- **Round toward** $-\infty$
  Always round *down*.

Examples for Rounding

Rounding Mode	+11.5	+12.5	-11.5	-12.5
to nearest, ties to even	+12.0	+12.0	-12.0	-12.0
to nearest, ties away from zero	+12.0	+13.0	-12.0	-13.0
toward 0	+11.0	+12.0	-11.0	-12.0
toward $+\infty$	+12.0	+13.0	-11.0	-12.0
toward $-\infty$	+11.0	+12.0	-12.0	-13.0

Comparing Floating Points Values

Many floating point operations are not associative due to rounding! That is, the same expression, but *computed in a different order,* may create different results!

This makes the comparison of floating point values particularly tricky and complicated.

Moreover, different compilers, even though implementing the same standard, may yield different results!

Problem: Absorbtion (C)

Absorption occurs when working with numbers with large differences in magnitude:

```c
#include <stdio.h>

int main(void) {
  float val1 = 100.0, val2 = 0.05, val3 = 0.05;

  printf("sum 1: %f\n", (val1 + val2) + val3);
  printf("sum 2: %f\n", val1 + (val2 + val3));
}
```

```
> ./absorbtion
sum 1 : 100.100006
sum 2 : 100.099998
```

Problem: Absorbtion (Java)

```
public class Absorbtion {
  public static void main (String args[]) {
    float val1 = 100.0f;
    float val2 = 0.05f;
    float val3 = 0.05f;

    System.out.println("sum 1 : " + ((val1 + val2) + val3));
    System.out.println("sum 2 : " + (val1 + (val2 + val3)));
    System.out.println("sum 3 : " + (val1 + val2 + val3));
  }
}
```

```
java Absorbtion
sum 1 : 100.100006
sum 2 : 100.1
sum 3 : 100.100006
```

Comparing Floats: Idea

⚠ Due to such problems, it is totally "forbidden" to compare floating point values using the == operator.

This solution — seen in many books — fails most of the time:

```
final float EPSILON = 10e-6;
...
if Math.abs(value1 - value2) < EPSILON {
  ...
}
```

Problem: The value of the constant `EPSILON` depends on the magnitude of the tested values.

Comparing Floats: Bitwise I

One can try to compare both values bit-by-bit. For example:

1. Transform the float values ⚠ bitwise into integers (e.g. in Java using the method `floatToIntBits()`)
2. Mask the 2 LSB of each value
3. Compare the result.

Question: ⚠ Why does this comparison fail?

Comparing Floats: Bitwise II

```
#include <stdio.h>
typedef union {
  int intVal;
  float floatVal;
} intFloat;

int main(void) {
  intFloat testVal;

  testVal.intVal = 0x3fffffff;
  printf("v1 as int : %x\n", testVal.intVal);
  printf("v1 as float : %f\n", testVal.floatVal);

  testVal.intVal = 0x40000001;
  printf("v2 as int : %x\n", testVal.intVal);
  printf("v2 as float : %f\n", testVal.floatVal);
  return 0;
}
```

Comparing Floats: Bitwise III

```
./a.out
v1 as int : 3fffffff
v1 as float : 2.000000
v2 as int : 40000001
v2 as float : 2.000000
```

Comparing Floats: Absolute Error I

- The main idea is to have an error margin which depends on the magnitude of the two values being compared.
- The simplest formula for that is:

$$\frac{|a-b|}{b} < \varepsilon$$

  for a given ε value independent of the magnitude of a and b.
- ⚠ If b is zero (or very near to zero) this direct computation fails!

Comparing Floats: Absolute Error II

```
diff ← $\big|\, |a| - |b| \,\big|$
if $a$ is strictly equal to $b$ then
  return true        {for $\infty$ and NaN}
else if $a = 0$ or $b = 0$ or ($|a| + |b| <$ min normal value) then
  {for very small value of $a$ and $b$}
  return diff $<$ ($\varepsilon *$ min normal value)
else
  return (diff / $\min(|a| + |b|$, max float value$)) < \varepsilon$;
end if
```

Note: *min normal value* is the smallest value which can be stored *without* using a denormalized value.

Total ordering predicate

The standard provides a predicate totalOrder which defines a total ordering for all floating point numbers in each format.

The predicate agrees with the normal comparison operations when they say one floating point number is less than another.

The normal predicate compares -0 and +0 as equal.

The normal comparison operations however treat NaNs as unordered.

Conclusion

Conclusion

Encoding Numbers

- Little and big endian,
- Least and most significant bytes (MSB/LSB),
- Signed numbers are encoded using 2's complement.

Real numbers

- Are composed of sign, mantissa and exponent.
- Equality ("==" does not work) and comparison is tricky.